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Two-kink bound states in the magnetically perturbed Potts field theory at $T < T_c$

S B Rutkevich

Institute of Solid State and Semiconductor Physics, SSPA 'Scientific-Practical Materials Research Centre, NAS of Belarus', P. Brovka St 17, 220072 Minsk, Belarus

E-mail: rut@ifttp.bas-net.by, rut57@mail.ru

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Abstract

The q-state Potts field theory with $2 \leqslant q \leqslant 4$ in the low-temperature phase is considered in the presence of a weak magnetic field h. In the absence of the magnetic field, the theory is integrable, but not free at $q > 2$: its elementary excitations—the kinks—interact at small distances, and their interaction can be characterized by the factorizable scattering matrix which was found by Chim and Zamolodchikov. The magnetic field induces long-range attraction between kinks causing their confinement into the bound states. We calculate the masses of the two-kink bound states in the leading order in $h \to \pm 0$ expressing them in terms of the scattering matrix of kinks at $h = 0$.

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1. Introduction

Kink topological excitations are quite common in two-dimensional field theories with Hamiltonian invariant under some discrete symmetry group $\mathcal{G}$. If such a symmetry is spontaneously broken in the ordered phase, the Hamiltonian has a discrete set of degenerate vacua $|0_\alpha\rangle$, $\alpha = 1, \ldots, q$. Then the kinks $K_{\alpha\beta}$, i.e. the domain walls separating two different vacua α and β, behave like stable quantum particles which can propagate in the system. Adding a small interaction, which explicitly breaks the Hamiltonian $\mathcal{G}$-symmetry, lifts the degeneracy of ground states $|0_\alpha\rangle$ and leads to confinement of kinks dividing the true and false vacua. This simple but quite general scenario of confinement in two dimensions originates to the work of McCoy and Wu [1]. Its particular realizations in different two-dimensional models have been the subject of considerable interest in recent years [2–8].

The simplest and most studied example of the two-dimensional model exhibiting confinement is the Ising field theory (IFT) [2], which is characterized by the $\mathbb{Z}_2$ symmetry group. At zero magnetic field, the theory describes free massive (apart from the critical point) neutral fermions, which represent in the low-temperature phase the kinks interpolating between two degenerate vacua. At small magnetic field $h > 0$, the kinks ('quarks') become confined into pairs which form a tower of bound states ('mesons') having the zero topological charge. The meson masses $M_n(h)$ densely fill the segment[1] $[2m, \infty)$ at $h \to 0$. Two asymptotic expansions describe $M_n(h)$ at small h in different regions of this segment.

(i) Near the edge point $2m$ (i.e. for fixed n at $h \to 0$) one can use the *low energy expansion* in the fractional powers of the magnetic field. Its leading term was obtained by McCoy and Wu [1], further corrections were found by Fonseca and Zamolodchikov [2, 9].

(ii) The masses of highly excited mesons (with $n \gg 1$, in particular for $n \sim 1/h$) are described by the *semiclassical expansion* in the integer powers of h [2, 10, 11].

Both the low energy and semiclassical expansions for the meson masses in the IFT were obtained by means of the perturbative analysis of the Bethe–Salpeter equation, first derived for this model by Fonseca and Zamolodchikov [9]. Since their original derivation procedure substantially exploited the free-fermionic structure of the IFT at $h = 0$, it was not applied to the models in which kinks interact at short distances already in the deconfined phase.

In this paper we address to the problem of extension of the Bethe–Salpeter approach to make it appropriate for calculation of meson masses in such two-dimensional models. The particular subject of our interest is the q-state Potts field theory (PFT) for $2 \leqslant q \leqslant 4$, which describes the scaling limit of the two-dimensional Potts model. In the model each site in the lattice has q different states ('colours'). At zero magnetic field, the Potts model is invariant under the group S_q of permutations of q colours. At $q = 2$, the Potts model reduces to the Ising model.

The zero-field PFT is integrable, i.e. it has infinite number of integrals of motion and the factorizable scattering matrix [12]. In the low-temperature phase, the particle content of the (zero-field) PFT contains $q(q-1)$ kinks $K_{\alpha\beta}, \alpha, \beta = 1, \ldots, q$, which interact with each other at short distances. Their S-matrix was found by Chim and Zamolodchikov [12].

The PFT in the presence of a nonzero magnetic field h acting on one of the q colours has been studied recently by Delfino and Grinza [5]. Beyond other results relating to the $T > T_c$ phase, these authors performed qualitative analysis of the kink confinement in this model, classified two-kink and three-kink bound states, and conjectured evolution of their mass spectra with temperature and magnetic field.

The main subject of our interest is the meson masses $M_n(h)$ in the q-state PFT in the low-temperature phase in the limit of the weak magnetic field $h \to 0$. As in [5], the magnetic field acting on one colour only is chosen. We consider the mesons as the bound states of two kinks, which attract one another with a linear potential at large distances, and undergo scattering upon collisions. As a result, we express the leading terms in both low energy and semiclassical expansions for $M_n(h)$ in terms of the known zero-field kink S-matrix.

The paper is organized as follows. In the next section, we recall the definition and some well-known properties of the Potts model on the square lattice and its scaling limit. Sections 3 and 4 describe the calculation of the meson masses in the leading order of the weak magnetic field in the low-temperature phase at $h \to +0$ and at $h \to -0$, respectively, for different values of q. In section 5, the Bethe–Salpeter equation for the PFT is derived. In the IFT, the analogous Bethe–Salpeter equation provides the basis for the systematic theory of mesons [2]. Concluding remarks are presented in section 6.

[1] Note that only the mesons having masses $M_n(h)$ below the two masses of the lightest meson are the stable particles: $M_n(h) < M_1(h) \approx 4m$. Heavier mesons are the resonances since they can decay into a pair of light mesons.

2. The Potts model and its scaling limit

In this section we describe briefly the definition and few basic properties of the q-state Potts model in two dimensions. We start from the square lattice model and then move to its scaling limit.

Consider the two-dimensional square lattice $\mathbb{Z}^2$, and associate the discrete spin variable $s(x) = 1, 2, \ldots, q$ with each lattice site $x \in \mathbb{Z}^2$. The Hamiltonian of the model is defined as

$$\mathcal{H} = -\frac{1}{T} \sum_{<x,y>} \delta_{s(x),s(y)} - H \sum_{x} \delta_{s(x),q}. \tag{1}$$

Here the first summation is over the nearest neighbour pairs, T is the temperature, H is the external magnetic field applied along the qth direction and $\delta_{\alpha,\alpha'}$ is the Kronecker symbol. At $H = 0$, Hamiltonian (1) is invariant under the permutation group S_q; at $H \neq 0$, the symmetry group reduces to S_{q-1}. By means of mapping onto the random cluster model [13, 14], one can also define the q-state Potts model with noninteger values of q.

The order parameters $\langle\sigma_\alpha\rangle$ can be associated with the variables

$$\sigma_\alpha(x) = \delta_{s(x),\alpha} - \frac{1}{q}, \qquad \alpha = 1, \ldots, q.$$

Parameters $\langle\sigma_\alpha\rangle$ are not independent, since

$$\sum_{\alpha=1}^{q} \sigma_\alpha(x) = 0. \tag{2}$$

The zero-field model undergoes the ferromagnetic phase transition at the critical temperature:

$$T_c = \frac{1}{\log(1+\sqrt{q})}. \tag{3}$$

This phase transition is first order for $q > 4$, and continuous for $2 \leqslant q \leqslant 4$. The ferromagnetic low-temperature phase at zero field is q-times degenerated. For a review of many other known properties of the Potts model see [15, 16].

We shall consider only the Potts model with $2 \leqslant q \leqslant 4$. In this case, the correlation length diverges at $H \to 0$, $T \to T_c$. The conformal field theory associated with this critical point is characterized by the central charge:

$$c(q) = 1 - \frac{6}{t(t+1)}, \qquad \text{where} \qquad \sqrt{q} = 2\sin\frac{\pi(t-1)}{2(t+1)}. \tag{4}$$

The scaling limit of model (1) is described by the action [5]

$$\mathcal{A} = \mathcal{A}_{\text{CFT}}^{(q)} - \tau \int \mathrm{d}^2x\, \varepsilon(x) - h \int \mathrm{d}^2x\, \sigma_q(x). \tag{5}$$

Here $\mathcal{A}_{\text{CFT}}^{(q)}$ corresponds to the conformal field theory, which is associated with the critical point. The fields $\varepsilon(x)$ (energy density) and $\sigma_q(x)$ (spin density) have the scaling dimensions

$$X_\epsilon^{(q)} = \frac{1}{2}\left(1 + \frac{3}{t}\right), \qquad X_\sigma^{(q)} = \frac{(t-1)(t+3)}{8t(t+1)}.$$

The couplings τ and h are proportional to the deviation of the temperature and magnetic field from their critical point values.

2.1. Ordered phase in the PFT at $h = 0$

The field theory (5) is integrable along the line $h = 0$ in the (τ, h)-plane. In this paper only the low-temperature ($\tau < 0$) phase will be considered. At $h = 0$ and $\tau < 0$, the S_q symmetry is spontaneously broken: the model has q degenerate vacua $|0_\alpha\rangle$, $\alpha = 1, 2, \ldots, q$, which are distinguished by the values of the order parameter:

$$\langle\sigma_\gamma\rangle_\alpha \equiv \langle 0_\alpha|\sigma_\gamma(x)|0_\alpha\rangle = \frac{v}{q-1}(q\,\delta_{\gamma,\alpha} - 1), \tag{6}$$

with some positive v. The symmetry group S_q acts by permutations of these vacua. Elementary excitations are the $q(q-1)$ kinks $K_{\alpha\beta}(\theta)$, which interpolate between different vacua α and β. Here θ denotes the kink rapidity, which parametrizes its energy and momentum

$$E = m\cosh\theta, \qquad p = m\sinh\theta, \tag{7}$$

with $m \sim |\tau|^{1/[2-X_\epsilon^{(q)}]}$ being the kink mass.

The two-kink scattering at $\tau < 0$, $h = 0$ is described by the Faddeev–Zamolodchikov commutation relations

$$K_{\alpha\gamma}(\theta_1)K_{\gamma\beta}(\theta_2) = \sum_{\delta\neq\alpha,\beta} S_{\alpha\beta}^{\gamma\delta}(\theta_{1,2})K_{\alpha\delta}(\theta_2)K_{\delta\beta}(\theta_1), \tag{8}$$

with the scattering amplitudes $S_{\alpha\beta}^{\gamma\delta}(\theta_{1,2})$, and $\theta_{12} = \theta_1 - \theta_2$. Due to the S_q invariance, only four scattering amplitudes are independent, providing

$$K_{\alpha\gamma}(\theta_1)K_{\gamma\beta}(\theta_2) = S_0(\theta_{12})\sum_{\delta\neq\gamma} K_{\alpha\delta}(\theta_2)K_{\delta\beta}(\theta_1) + S_1(\theta_{12})K_{\alpha\gamma}(\theta_2)K_{\gamma\beta}(\theta_1), \qquad \alpha\neq\beta, \tag{9}$$

$$K_{\alpha\gamma}(\theta_1)K_{\gamma\alpha}(\theta_2) = S_2(\theta_{12})\sum_{\delta\neq\gamma} K_{\alpha\delta}(\theta_2)K_{\delta\alpha}(\theta_1) + S_3(\theta_{12})K_{\alpha\gamma}(\theta_2)K_{\gamma\alpha}(\theta_1). \tag{10}$$

The explicit expressions for the scattering amplitudes were determined in [12]:

$$S_0(\theta) = \frac{\sinh\lambda\theta\,\sinh\lambda(\theta - \mathrm{i}\pi)}{\sinh\lambda\left(\theta - \frac{2\pi\mathrm{i}}{3}\right)\sinh\lambda\left(\theta - \frac{\mathrm{i}\pi}{3}\right)}\Pi\left(\frac{\lambda\theta}{\mathrm{i}\pi}\right), \tag{11}$$

$$S_1(\theta) = \frac{\sin\frac{2\pi\lambda}{3}\sinh\lambda(\theta - \mathrm{i}\pi)}{\sin\frac{\pi\lambda}{3}\sinh\lambda\left(\theta - \frac{2\mathrm{i}\pi}{3}\right)}\Pi\left(\frac{\lambda\theta}{\mathrm{i}\pi}\right), \tag{12}$$

$$S_2(\theta) = \frac{\sin\frac{2\pi\lambda}{3}\sinh\lambda\theta}{\sin\frac{\pi\lambda}{3}\sinh\lambda\left(\theta - \frac{\mathrm{i}\pi}{3}\right)}\Pi\left(\frac{\lambda\theta}{\mathrm{i}\pi}\right), \tag{13}$$

$$S_3(\theta) = \frac{\sin\lambda\pi}{\sin\frac{\pi\lambda}{3}}\Pi\left(\frac{\lambda\theta}{\mathrm{i}\pi}\right). \tag{14}$$

The parameter λ is related to q as

$$\sqrt{q} = 2\sin\frac{\pi\lambda}{3}, \tag{15}$$

and

$$\Pi\left(\frac{\lambda\theta}{\mathrm{i}\pi}\right) = \frac{\sinh\lambda\left(\theta + \mathrm{i}\frac{\pi}{3}\right)}{\sinh\lambda(\theta - \mathrm{i}\pi)}\mathrm{e}^{\mathcal{A}(\theta)}, \tag{16}$$

$$\mathcal{A}(\theta) = \int_0^\infty \frac{\mathrm{d}x}{x}\frac{\sinh\frac{x}{2}\left(1 - \frac{1}{\lambda}\right) - \sinh\frac{x}{2}\left(\frac{1}{\lambda} - \frac{5}{3}\right)}{\sinh\frac{x}{2\lambda}\cosh\frac{x}{2}}\sinh\frac{x\theta}{\mathrm{i}\pi}. \tag{17}$$

Note that the values of the parameter λ corresponding to the integer q are

$$\lambda = \tfrac{3}{4} \qquad \text{for } q = 2,$$
$$\lambda = 1 \qquad \text{for } q = 3,$$
$$\lambda = \tfrac{3}{2} \qquad \text{for } q = 4.$$

2.2. Kink confinement in a weak magnetic field

Application of a small magnetic field along the q-direction lifts degeneracy between the vacuum $|0_q\rangle$ and the vacua $|0_\alpha\rangle$ with $\alpha < q$. In the first order in h, the shift $\Delta\mathcal{E}$ between their energy densities reads

$$\Delta\mathcal{E} = \delta\mathcal{E}_\alpha - \delta\mathcal{E}_q = \frac{vq}{q-1}\, h \qquad \text{for} \qquad \alpha = 1, \dots, q-1. \tag{18}$$

It gives rise to the linear attractive potential between two kinks which interpolate between the stable and false vacua,

$$V(x_1, x_2) = (x_2 - x_1)\Delta\mathcal{E}, \tag{19}$$

where $x_1 < x_2$ are the spacial coordinates of the kinks.

Depending on the sign of h, two regimes are distinguished [5].

- If $h > 0$, the vacuum $|0_q\rangle$ becomes the true ground state of the system, and the states $|0_\alpha\rangle$ with $\alpha \neq q$ become the false vacua. The magnetic field induces a long-range attraction between kinks leading to their confinement. Isolated kinks do not survive as asymptotic states of the theory, and the elementary excitations are the bound states of two and three kinks[2].
- If $h < 0$, the vacuum $|0_q\rangle$ becomes metastable, and the true vacuum states $|0_\alpha\rangle$ with $\alpha = 1, \dots, q-1$ are still degenerate in the energy. Elementary excitations are the kinks $K_{\alpha\beta}(\theta)$ interpolating between the true vacua $\alpha, \beta \neq q$. On the other hand, the kinks $K_{\alpha q}(\theta_1)$ and $K_{q\beta}(\theta_2)$ are confined into the bound states by the magnetic field. However, such bound states are unstable due to decay into isolated kinks $K_{\alpha\beta}(\theta)$ (see [5] and the discussion below in section 4).

Note that the kinks in the fields theories with confinement are often called 'quarks', while their bound states play the role of 'mesons' (kink–antikink states), and 'baryons' (tree-kink states).

3. Meson masses at $h \to +0$

The meson mass M can be formally determined from the solution of the eigenvalue problem:

$$\hat{\mathcal{H}}(h)|\pi(P)\rangle = [E(P) + E_{\text{vac}}]|\pi(P)\rangle, \tag{20}$$

$$\hat{P}|\pi(P)\rangle = P|\pi(P)\rangle, \tag{21}$$

where $\hat{\mathcal{H}}(h)$ is the Hamiltonian, $\hat{P}$ is the total momentum operator corresponding to action (5), E_{vac} is the vacuum energy, and $E(P)$ is the meson energy spectrum, which should have the relativistic form

$$E(P) = (P^2 + M^2)^{1/2}. \tag{22}$$

[2] The three-kink bound states can exist if $3 \leqslant q \leqslant 4$. The four-kink bound states, which are allowed for $q = 4$, should be unstable due to their decay into a pair of mesons [5].

Unfortunately, the explicit form of the PFT Hamiltonian $\hat{\mathcal{H}}(h)$ is known only in the case $q = 2$, which corresponds to the IFT. Below we describe briefly how the meson masses at $q = 2$ can be calculated from the perturbative analysis of the Bethe–Salpeter equation in the coordinate representation [9, 10]. This procedure will be then naturally generalized to the case $2 < q \leqslant 4$.

3.1. Ising field theory case: $q = 2$

In the IFT, kinks are fermions which are free at $h = 0$. For nonzero h, the Hamiltonian $\hat{\mathcal{H}}(h)$ can be written as

$$\hat{\mathcal{H}}(h) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}\, \omega(p)\, a_p^\dagger\, a_p - h \int_{-\infty}^{\infty} \mathrm{d}x\, \sigma_2(x), \tag{23}$$

where $\omega(p) = \sqrt{p^2 + m^2}$ is the spectrum of free fermions, and formfactors of the spin operator $\sigma_2(x)$ are explicitly known [17]. At small h, one can treat the meson as a bound state of two quarks neglecting four-quark, six-quark, etc contributions in its wavefunction. This two-quark approximation is asymptotically exact in the leading order in $h \to 0$. In this approximation, the meson mass M_n can be calculated from the perturbative solution of the Bethe–Salpeter equation. In the coordinate representation, the equation written in the meson rest frame reads

$$2\omega(\hat{p})\, \phi^{(n)}(x) + \Delta\mathcal{E}|x|\, \phi^{(n)}(x) + \Delta\mathcal{E}\, \hat{U}\, \phi^{(n)}(x) = M_n\, \phi^{(n)}(x), \tag{24}$$

$$\text{where} \qquad -\infty < x < \infty.$$

Here $\Delta\mathcal{E} = 2h\langle\sigma_2\rangle$ is the 'string tension', $\langle\sigma_2\rangle$ is the spontaneous magnetization at zero field, $|x|$ is the distance between the two quarks, $\hat{p} = -\mathrm{i}\partial_x$, and $\phi^{(n)}(x)$ denotes the configuration–space wavefunction in the two-quark approximation:

$$\phi^{(n)}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}\, \mathrm{e}^{\mathrm{i}px}\, \langle 0|a_{(P-p)}a_p|\pi_n(P)\rangle_{P\to 0}, \tag{25}$$

$$\phi^{(n)}(-x) = -\phi^{(n)}(x). \tag{26}$$

The first and the second terms on the left-hand side of (24) correspond to the kinetic energy of two quarks, and to their long-range attraction, respectively. The linear integral operator $\hat{U}$ describes the the short-range interaction between quarks in the two-fermion sector; its kernel $U(x, x')$ exponentially vanishes at large distances $|x| \gg m^{-1}$.

Several perturbative schemes have been developed for the IFT Bethe–Salpeter equation. The most convenient for us is the procedure described in [10]. It is based on the observation that at small h the average distance between the quarks in the meson is large compared with the correlation length m^{-1}. Therefore, in the right 'transport region' at $x \gg m^{-1}$, one can very well approximate the solution $\phi^{(n)}(x)$ of the Bethe–Salpeter equation (24) by the function $\Phi^{(n)}(x)$, which is bounded in the whole line $-\infty < x < \infty$ and solves the equation:

$$[2\omega(\hat{p}) - M_n + \Delta\mathcal{E}x]\Phi^{(n)}(x) = 0. \tag{27}$$

After the Fourier transform this equation takes the form

$$[2\omega(p) - M_n + \mathrm{i}\Delta\mathcal{E}\partial_p]\Phi^{(n)}(p) = 0, \tag{28}$$

providing

$$\Phi^{(n)}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \exp\left\{ \frac{\mathrm{i}[f(p) - pM_n]}{\Delta\mathcal{E}} + \mathrm{i}px \right\}, \tag{29}$$

where

$$f(p) = 2\int_0^p \mathrm{d}q\, \omega(q) = m^2\left[\theta + \frac{\sinh 2\theta}{2}\right], \tag{30}$$

and $p = m\,\sinh\theta$.

In the left 'transport region' $x < 0$, $|x| \gg m^{-1}$, the function $\phi^{(n)}(x)$ should approach to $-\Phi^{(n)}(-x)$ due to (26). In the intermediate 'scattering region' $|x| \lesssim m^{-1}$, one can solve equation (24) perturbatively in h. Joining solutions obtained in these three regions leads to the condition which gives the meson mass spectrum $M_n(h)$.

It turns out that *in the leading order in h*, the meson masses can be obtained from the equation

$$\Phi^{(n)}(x)|_{x=0} = 0, \tag{31}$$

since the odd continuous function $\phi^{(n)}(x)$ can be approximated (in the leading order in h) by integral (29) in the whole positive half-axis $x > 0$.

At $x = 0$, $M_n > 2$ and $h \to +0$, the integral in (29) is determined by two saddle points $\pm p_n$, where

$$p_n = m\sinh\beta_n, \tag{32}$$

and β_n parametrizes the meson masses M_n:

$$M_n = 2m\cosh\beta_n. \tag{33}$$

In the leading order in h, this yields the first term of the semiclassical expansion [2, 10]

$$\sinh(2\beta_n) - 2\beta_n = 2\pi\left(n - \frac{1}{4}\right)\zeta + O(\zeta^2), \tag{34}$$

where $\zeta = \Delta\mathcal{E}/m^2 \sim h$. The semiclassical expansion holds if $\zeta \ll 1$ and $n \gg 1$.

If M_n approaches $2m$, two saddle points $\pm p_n$ merge at the origin, and integral (29) becomes proportional to the Airy function:

$$\begin{aligned}\Phi^{(n)}(x)|_{x=0} &= m\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{2\pi}\exp\left\{\frac{\mathrm{i}}{\zeta}\left[\frac{\theta^3}{3} - \theta\frac{(M_n-2m)}{m}\right]\right\} + O(\zeta)\\ &= m\zeta^{1/3}\mathrm{Ai}\left[-\frac{(M-2m)}{m\zeta^{2/3}}\right] + O(\zeta)\end{aligned} \tag{35}$$

in the limit $h \to 0$. Formulae (31) and (35) give rise to the leading term of the low-energy expansion [1]:

$$\frac{M_n}{m} - 2 = z_n\zeta^{2/3} + O(\zeta^{4/3}), \tag{36}$$

where $-z_n$ denotes the nth zero of the Airy function, $\mathrm{Ai}(-z_n) = 0$, $n = 1, 2, \ldots.$ The low-energy expansion holds if $n\zeta \ll 1$. It corresponds to the nonrelativistic approximation in the quark dispersion law $\omega(\hat{p}) = m + \hat{p}^2/(2m) + \cdots$ in equation (27).

3.2. Generic q

Let us discuss now how the procedure described above should be modified to be applied to the PFT with $2 < q \leqslant 4$, or more generally, to the models exhibiting weak confinement at small h, which are integrable but not free at $h = 0$. In this subsection the parameter q is allowed to take fractional values, which are assumed not to be too close to 3.

First, it is natural to expect that the two-quark approximation can be safely used at small h, if the distance between quarks is much larger than the correlation length m^{-1}. We adopt two further assumptions.

(i) At large distances $x_2 - x_1 \gg m^{-1}$, the interaction between the two quarks $K_{q\alpha}(x_1)$ and $K_{\alpha q}(x_2)$ forming a meson is completely described by the linear attractive potential:

$$V(x_1, x_2) = (x_2 - x_1)\Delta\mathcal{E} \tag{37}$$

with the string tension

$$\Delta\mathcal{E} = \frac{vq}{q-1}h + o(h). \tag{38}$$

(ii) To the linear order in the magnetic field, the quark dispersion law $\epsilon(p; h)$ is the same as the free-fermion spectrum $\omega(p) = (p^2 + m^2)^{1/2}$:

$$\epsilon(p; h) = \omega(p) + o(h). \tag{39}$$

These natural assumptions summarize the experience gained from the IFT where they can be verified by means of the consistent perturbation theory based on the Bethe–Salpeter equation [2, 11]. In particular, radiative corrections to the quark string tension and dispersion law in the IFT are known to be of the third and second order in h, respectively. Furthermore, the approach based on the above assumptions allows one to reproduce in a simple way the leading order of the exact asymptotical expansions for the meson masses in the IFT, see section 2 of [2].

The Bethe–Salpeter equation (90) for the q-state PFT will be derived in section 5. The kernel of this integral equation is expressed in the matrix element (94) of the magnetization operator between the two-quark states with definite rapidities. Since the explicit form of this matrix element is not known for $q > 2$, a direct proof of statements (i) and (ii) for general PFT is still impossible, and we shall take them as assumptions.

A zero-momentum meson state $\pi(0)$ can be characterized by the $(q-1)$-component wavefunction $\psi_\alpha(x)$:

$$\psi_\alpha(x) = \langle K_{q\alpha}(x_1 + x)K_{\alpha q}(x_1) \mid \pi(0)\rangle, \qquad \alpha = 1, \ldots, q-1, \tag{40}$$

which is well defined for large positive $x \gg m^{-1}$.

Action (5) and the true vacuum $|0_q\rangle$ are invariant under the group S_{q-1} of permutations of the first $q-1$ colours. In what follows, we shall use the results of the S_{q-1} symmetry analysis of the meson states, which has been done by Delfino and Grinza [5]. The meson states form $(q-1)$ multiplets π_k. Here π_k are the eigenstates of the generator Ω_{q-1} of cyclic permutations of first $q-1$ colours $\alpha = 1, \ldots, q-1$,

$$\Omega_{q-1}\pi_k = \gamma^k \pi_k, \tag{41}$$

$$\text{where} \qquad \gamma = \exp[2\pi \mathrm{i}/(q-1)], \qquad k = 0, \ldots, q-2.$$

This symmetry determines the α-dependence of the wavefunction $\psi_{k,\alpha}(x)$ for the state $\pi_k(0)$,

$$\psi_{k,\alpha}(x) = \gamma^{-k\alpha}\phi_k(x). \tag{42}$$

Due to (37) and (39), the meson wavefunction $\phi_k^{(n)}(x)$ corresponding to the state $\pi_k^{(n)}(0)$ should satisfy equation (27) at $x \gg m^{-1}$, and therefore, we can write it in the form

$$\phi_k^{(n)}(x) \approx \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \exp\left\{ \frac{\mathrm{i}[f(p) - pM_n]}{\Delta\mathcal{E}} + \mathrm{i}px \right\} \qquad \text{for} \qquad x \gg m^{-1}, \tag{43}$$

where $f(p)$ is given by (30).

The further analysis depends on the value of n.

3.2.1. Semiclassical case For large $n \gg 1$ and fixed $x \gtrsim m^{-1}$, the $h \to 0$ asymptotics of integral (43) is determined by the contributions of two saddle points (32), providing

$$\phi_k^{(n)}(x) \approx \left(\frac{\Delta\mathcal{E}}{4\pi \tanh\beta_n}\right)^{1/2} \left[C_k(\beta_n)\, \mathrm{e}^{\mathrm{i}mx\sinh\beta_n} + C_k^*(\beta_n)\, \mathrm{e}^{-\mathrm{i}mx\sinh\beta_n}\right],$$

$$\text{with} \qquad C_k(\beta_n) = \exp\left[\frac{\mathrm{i}\left(\beta_n - \frac{\sinh 2\beta_n}{2}\right)}{\zeta} + \frac{\mathrm{i}\pi}{4}\right]. \tag{44}$$

Here again parametrization (33) is implied. The coefficients $C_k^*(\beta_n)$ and $C_k(\beta_n)$ are just the in and out amplitudes, which characterize the quark 'plane waves' which enter and leave at $x \sim m^{-1}$ the scattering region $0 < x \lesssim m^{-1}$.

At zero order in h, these amplitudes should be related by the scattering condition [5]

$$C_k(\beta_n) = \{S_2(2\beta_n)[(q-1)\delta_{k,0} - 1] + S_3(2\beta_n)\}C_k^*(\beta_n). \tag{45}$$

The explicit form of the factor in the curly brackets on the right-hand side is

$$(q-2)S_2(\theta) + S_3(\theta) = -\mathrm{e}^{\mathcal{A}(\theta)} \frac{\sinh\left[\lambda\left(\frac{\mathrm{i}\pi}{3}+\theta\right)\right]}{\sinh\left[\lambda\left(\frac{\mathrm{i}\pi}{3}-\theta\right)\right]} \frac{\sinh[\lambda(\mathrm{i}\pi+\theta)]}{\sinh[\lambda(\mathrm{i}\pi-\theta)]}, \quad \text{for} \quad k = 0, \tag{46}$$

$$S_3(\theta) - S_2(\theta) = -\mathrm{e}^{\mathcal{A}(\theta)} \frac{\sinh\left[\lambda\left(\frac{\mathrm{i}\pi}{3}+\theta\right)\right]}{\sinh\left[\lambda\left(\frac{\mathrm{i}\pi}{3}-\theta\right)\right]}, \quad \text{for} \quad k = 1, \ldots, q-2. \tag{47}$$

Equations (44)–(46) lead to the semiclassical quantization condition:

$$\sinh(2\beta_n) - 2\beta_n = \left[2\pi\left(n - \tfrac{1}{4}\right) + \mathrm{i}\mathcal{A}(2\beta_n) + 2\alpha_1(2\beta_n) + 2\alpha_2(2\beta_n)\right]\zeta + O(\zeta^2), \tag{48}$$

where

$$\alpha_1(\theta) = \arctan\left[\tanh(\lambda\theta)\cot(\pi\lambda)\right], \qquad \alpha_2(\theta) = \arctan\left[\tanh(\lambda\theta)\cot(\pi\lambda/3)\right]. \tag{49}$$

Equations (33), (48) determine the masses of π_0 for $n \gg 1$.

For the multiplet π_k, $k = 1, \ldots, q-2$, the analogous quantization condition

$$\sinh(2\beta_n) - 2\beta_n = \left[2\pi\left(n - \frac{1}{4}\right) + \mathrm{i}\mathcal{A}(2\beta_n) + 2\alpha_2(2\beta_n)\right]\zeta + O(\zeta^2) \tag{50}$$

follows from equations (44), (45), and (47).

For $q = 4$, the masses of π_0 and π_k, $k = 1, \ldots, q-2$, are the same:

$$\sinh(2\beta_n) - 2\beta_n = \left[2\pi\left(n - \frac{1}{4}\right) + \mathrm{i}\mathcal{A}(2\beta_n)\right]\zeta + O(\zeta^2) \quad \text{for} \quad k = 0, \ldots, q-2. \tag{51}$$

3.2.2. Low-energy expansion Consider the leading order of the low-energy expansion $h \to +0$, $n \sim 1$. The functions $\phi_k(x)$ are smooth in this case and vary on the scale much larger than the correlation length m^{-1}. At large positive $x \gg m^{-1}$, they should satisfy the differential equation

$$\left[2m - M_n - \frac{1}{m}\partial_x^2 + x\Delta\mathcal{E}\right]\phi_k(x) = 0. \tag{52}$$

The solution regular at $x \to \infty$ is given by the Airy function

$$\phi_k(x) = \mathrm{Ai}(t - t_n), \tag{53}$$

where

$$t = (m\Delta\mathcal{E})^{1/3}x, \qquad t_n = \frac{(M_n - 2m)m^{1/3}}{(\Delta\mathcal{E})^{2/3}}. \tag{54}$$

For noninteger λ equations (46), (47) yield

$$\lim_{\theta\to 0}\{S_2(\theta)[(q-1)\delta_{k,0} - 1] + S_3(\theta)\} = -1, \tag{55}$$

$$\lim_{\theta\to 0}\{S_3(\theta) - S_2(\theta)\} = -1. \tag{56}$$

This implies the fermionic boundary condition for the wavefunction $\phi_k(0) = 0$ for $k = 0, \ldots, q-2$ leading to the same low-energy mass spectrum (36) for all mesons π_k.

3.3. *Weak coupling expansion for $q = 3$*

A separate consideration is needed at $q = 3$. Relations (13)–(17) reduce in this case to

$$\lambda = 1, \tag{57}$$

$$S_2(\theta) = -\frac{\sinh(\theta + \mathrm{i}\pi/3)}{\sinh(\theta - \mathrm{i}\pi/3)}\,\mathrm{e}^{\mathcal{A}(\theta)}, \qquad S_3(\theta) = 0, \tag{58}$$

$$\mathcal{A}(\theta) = \int_0^\infty \frac{\mathrm{d}x}{x}\frac{2\sinh(x/3)}{\sinh x}\sinh\frac{x\theta}{\mathrm{i}\pi}. \tag{59}$$

The scattering condition (45) now takes the form

$$C_k(\beta_n) = (-1)^k S_2(2\beta_n)C_k^*(\beta_n). \tag{60}$$

In the low-energy case, the rapidities of quarks are small, $\theta_{12} \ll 1$, and

$$S_2(\theta_1 - \theta_2) \approx S_2(0) = 1. \tag{61}$$

Thus, the boundary conditions for equation (52) should be bosonic for $\phi_0(x)$,

$$\phi_0'(0) = 0, \tag{62}$$

and fermionic for $\phi_1(x)$,

$$\phi_1(0) = 0. \tag{63}$$

Therefore, whereas the mass spectrum of π_1 in the low-energy region is still given by equation (36), the masses of π_0 are described now by

$$\frac{M_n}{m} - 2 = \zeta^{2/3}z_n' + O(\zeta^{4/3}), \tag{64}$$

with $(-z_n')$ being the zeros of the first derivative of the Airy function $\mathrm{Ai}'(-z_n') = 0$, $n = 1, 2, \ldots$.

It is tempting to associate this peculiar behaviour of the low-energy π_0 mass spectrum in the $q = 3$ case with the presence of the B-meson in the zero-field PFT [12]. At $q = 3$ and $h \to +0$, its mass M_B lies exactly at the edge point $2m$ of the π-meson spectra.

At large $n \gg 1$, the zeros of the Airy function and its derivative behave as [18]

$$z_n \approx \left[\frac{3\pi(4n-1)}{8}\right]^{2/3}, \qquad z_n' \approx \left[\frac{3\pi(4n-3)}{8}\right]^{2/3}. \tag{65}$$

Combining these asymptotics with (36), (64), and (33), we obtain

$$\frac{(2\beta_n)^3}{3!} \approx 2\pi\left(n-\frac{3}{4}\right)\zeta, \qquad \text{for } k=0, \tag{66}$$

$$\frac{(2\beta_n)^3}{3!} \approx 2\pi\left(n-\frac{1}{4}\right)\zeta, \qquad \text{for } k=1, \tag{67}$$

at $n \gg 1$.

In the semiclassical region, quantization conditions (48) and (50) reduce at $q = 3$ to the form

$$\sinh(2\beta_n) - 2\beta_n = \left[2\pi\left(n-\frac{3}{4}\right) + 2\arctan\left(\frac{\tanh 2\beta_n}{\sqrt{3}}\right) + \mathrm{i}\mathcal{A}(2\beta_n)\right]\zeta + O(\zeta^{4/3}) \tag{68}$$

for $k = 0$, and

$$\sinh(2\beta_n) - 2\beta_n = \left[2\pi\left(n-\frac{1}{4}\right) + 2\arctan\left(\frac{\tanh 2\beta_n}{\sqrt{3}}\right) + \mathrm{i}\mathcal{A}(2\beta_n)\right]\zeta + O(\zeta^{4/3}) \tag{69}$$

for $k = 1$. At $\beta_n \to 0$ these relations agree with (66) and (67).

4. Meson masses at $h \to -0$

In negative magnetic field orientated along the qth direction, the kinks $K_{\alpha q}$ and $K_{q\beta}$ interpolating between the true and the false vacua become confined, while the kinks $K_{\alpha\beta}$ connecting two true vacua remain stable. Coupling two attracting kinks into bound states, one could construct the meson states both in the topological charged and topological neutral sectors.

In the topological charged sector, the meson state $\pi_{\alpha\beta}(0)$ in the 'transport' region $x_2 - x_1 > a/m$, with some constant $a \gg 1$, can be written as

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{x_1+a/m}^{\infty} \mathrm{d}x_2 |K_{\alpha q}(x_1) K_{q\beta}(x_2)\rangle \psi_{\alpha\beta}(x_2 - x_1), \tag{70}$$

where the meson wavefunction $\psi_{\alpha\beta}(x)$ should satisfy equation (27) with the string tension

$$\Delta\mathcal{E} = |h|(\langle 0_q|\sigma_q|0_q\rangle - \langle 0_\alpha|\sigma_q|0_\alpha\rangle) = \frac{q}{q-1}|h|v, \qquad \alpha \neq q. \tag{71}$$

However, the two kinks can become deconfined after the scattering process

$$K_{\alpha q}(\theta_1) K_{q\beta}(\theta_2) \to K_{\alpha\gamma}(\theta_2) K_{\gamma\beta}(\theta_1), \qquad \gamma \neq q, \tag{72}$$

characterized by the amplitude $S_0(\theta_{12})$ in (9). As a result, the topologically charged mesons $\pi_{\alpha\beta}(P)$ are already unstable in the leading order in h for $q > 3$.

On the other hand, the decay channel (72) is evidently closed for $q = 3$. For the state $K_{13}(\theta_1)K_{32}(\theta_2)$, the remaining scattering process

$$K_{13}(\theta_1)K_{32}(\theta_2) = S_1(\theta_{12})K_{13}(\theta_2)K_{32}(\theta_1) \tag{73}$$

is characterized by amplitude (12), which reduces at $q = 3$ to the form

$$S_1(\theta) = -e^{\mathcal{A}(\theta)}. \tag{74}$$

Reproducing, with minimal changes, calculations described in section 3, we obtain the masses of the topologically charged mesons π_{12} for $q = 3$. For large $n \gg 1$, they are described by the semiclassical quantization condition:

$$\sinh(2\beta_n) - 2\beta_n = \left[2\pi\left(n - \frac{1}{4}\right) + \mathrm{i}\mathcal{A}(2\beta_n)\right]\zeta + O(\zeta^2), \tag{75}$$

and equation (33). For small n, $n \ll \zeta^{-1}$, the masses of π_{12} are described by equation (36). Note that $\zeta = \Delta\mathcal{E}/m^2 \sim |h|$ at negative h.

In the topological neutral sectors, mesons $\pi_{\alpha\alpha}$ at $q \neq 4$ can easily decay due to decoupling of kinks in the process

$$K_{\alpha q}(\theta_1)K_{q\alpha}(\theta_2) \to K_{\alpha\gamma}(\theta_2)K_{\gamma\alpha}(\theta_1), \qquad \gamma \neq q. \tag{76}$$

In zero order in h, the scattering amplitude $S_2(\theta_{12})$ of this channel is given by equation (13).

The kink decoupling is hindered at $q = 4$, since $S_2(\theta_{12})$ vanishes in this case. As a result, commutation relation (14) for the mutual scattering of quarks $K_{\alpha q}(\theta_1)$ and $K_{q\alpha}(\theta_2)$ reduces at $q = 4$ to the form

$$K_{\alpha 4}(\theta_1)K_{4\alpha}(\theta_2) = S_3(\theta_{12})K_{\alpha 4}(\theta_2)K_{4\alpha}(\theta_1), \tag{77}$$

where $\alpha = 1, 2, 3$, and

$$S_3(\theta) = -e^{\mathcal{A}(\theta)}. \tag{78}$$

Accordingly, the masses of $\pi_{\alpha\alpha}$ are described at $q = 4$ by relations (36) and (75) in the leading order in h. Note that the function $\mathcal{A}(\theta)$ also depends on the parameter q through its (not indicated explicitly) dependence on λ, see equation (17). It is natural to expect that channel (76) opens in higher orders in h making all the mesons unstable in the topologically neutral sector at $h \to -0$ and $q = 4$.

5. Bethe–Salpeter equation

The heuristic approach applied in this paper is based on the assumptions adopted in subsection 3.2. We postulate the simple form (37) of the interaction between quarks at large distances, take the quark dispersion law in the form (39), and apply the boundary condition (45) originating from the scattering matrix at $h = 0$. Though this procedure seems to be sufficient for determining the leading order of the meson masses at $|h| \to 0$, it is desirable to have a more systematic theory suitable for the verification of the assumptions adopted in subsection 3.2, and for the calculation of higher order corrections to M_n. In the IFT, an efficient technique based on the Bethe–Salpeter equation has been developed by Fonseca and Zamolodchikov [2]. In this section we describe how a similar Bethe–Salpeter equation for the PFT can be derived. We hope that it will be used in the future for more consequent calculation of the weak-coupling expansion of the meson masses in the PFT.

The meson energy spectra $E(P)$ are determined by the eigenvalue problem (20) and (21), which we rewrite as

$$\hat{\mathcal{H}}_0 \,|\pi(P)\rangle - h\int_{-\infty}^{\infty} \mathrm{d}x\, \sigma_q(x)|\pi(P)\rangle = [E(P) + E_{\mathrm{vac}}]\, |\pi(P)\rangle,$$

$$\hat{P}\,|\pi(P)\rangle = P\,|\pi(P)\rangle. \tag{79}$$

Here the Hamiltonian $\hat{\mathcal{H}}_0$ corresponds to the integrable zero-field PFT in the ordered phase. Integrability of PFT at $h = 0$ implies, in particular, that the Hamiltonian $\mathcal{H}_0$ (together with the momentum operator $\hat{P}$) can be diagonalized by the multi-kink states $|K_{\underline{\alpha}}(\underline{p})\rangle$:

$$\hat{\mathcal{H}}_0|K_{\underline{\alpha}}(\underline{p})\rangle = [\omega(p_1) + \cdots + \omega(p_n)]\, |K_{\underline{\alpha}}(\underline{p})\rangle, \tag{80}$$

$$\hat{P}\,|K_{\underline{\alpha}}(\underline{p})\rangle = (p_1 + \cdots + p_n)\,|K_{\underline{\alpha}}(\underline{p})\rangle, \tag{81}$$

where

$$|K_{\underline{\alpha}}(\underline{p})\rangle = |K_{\alpha_0,\alpha_1}(p_1)K_{\alpha_1,\alpha_2}(p_2)\cdots K_{\alpha_{n-1},\alpha_n}(p_n)\rangle$$

$$= \frac{|K_{\alpha_0,\alpha_1}(\theta_1)K_{\alpha_1,\alpha_2}(\theta_2)\cdots K_{\alpha_{n-1},\alpha_n}(\theta_n)\rangle}{[\omega(\theta_1)\cdots\omega(\theta_n)]^{1/2}}, \tag{82}$$

$\alpha_{j+1} \neq \alpha_j$, the kink momenta are ordered as $\infty > p_1 > p_2 > \cdots > p_n > -\infty$, and $\theta_j = \operatorname{arcsinh}(p_j/m)$ are the corresponding rapidity variables. The kink states in the momentum and rapidity bases are normalized as

$$\langle K_{\beta,\alpha}(p)|K_{\alpha,\beta}(p')\rangle = 2\pi\delta(p - p'),$$

$$\langle K_{\beta,\alpha}(\theta)|K_{\alpha,\beta}(\theta')\rangle = 2\pi\delta(\theta - \theta').$$

Note that the particle sector of the PFT at $3 < q \leqslant 4$, $h = 0$, also contains the topologically neutral kink–antikink bound states $B(p)$, see [12]. Of course, the mesons $B(p)$ can appear together with kinks in the asymptotical in- and out-states at $h = 0$. We do not display the mesons $B(p_j)$ explicitly in formulae (80)–(83) just to avoid too cumbersome notations.

Let us now turn to equations (79) concentrating on the case of a positive magnetic field, $h > 0$. Then the meson vector $|\pi(P)\rangle$ being a topologically neutral state in the sector q should admit the expansion

$$|\pi(P)\rangle = \sum_{n=2}^{\infty}\ \sum_{\alpha_1,\ldots,\alpha_{n-1}\neq q}\ \int_{\infty>p_1>\cdots>p_n>-\infty} \frac{\mathrm{d}p_1\cdots\mathrm{d}p_n}{(2\pi)^n}$$

$$\cdot|K_{q,\alpha_1}(p_1)K_{\alpha_1,\alpha_2}(p_2)\cdots K_{\alpha_{n-1},q}(p_n)\rangle\langle K_{q,\alpha_{n-1}}(p_n)\cdots K_{\alpha_2,\alpha_1}(p_2)K_{\alpha_1,q}(p_1)|\pi(P)\rangle. \tag{83}$$

In complete analogy with the IFT, the two-quark approximation is based on the assumption that at $h \to +0$, the first term with $n = 2$ dominates in the infinite sum over n in (83). Accordingly, in the two-quark approximation one replaces the exact eigenvalue problem (79) by its projection onto the two-quark subspace $\mathbf{H}_2^{(q)}$ spanned by the basis $|K_{q,\alpha}(p_1)K_{\alpha,q}(p_2)\rangle$, with $\alpha \neq q$ and $p_1 > p_2$:

$$\mathcal{H}_0\,|\tilde{\pi}(P)\rangle - h\int_{-\infty}^{\infty} \mathrm{d}x\, \mathcal{P}_2^{(q)}\sigma_q(x)|\tilde{\pi}(P)\rangle = [\tilde{E}(P) + \tilde{E}_{\mathrm{vac}}]\,|\tilde{\pi}(P)\rangle,$$

$$\hat{P}\,|\tilde{\pi}(P)\rangle = P\,|\tilde{\pi}(P)\rangle, \tag{84}$$

where $|\tilde{\pi}(P)\rangle \in \mathbf{H}_2^{(q)}$, and $\mathcal{P}_2^{(q)}$ is the orthogonal projector on $\mathbf{H}_2^{(q)}$. Tildes distinguish solutions of (84) from those of the exact eigenvalue problem (79). The meson state in this approximation is characterized by the two-quark wavefunction $\Psi_\alpha(p_1, p_2)$:

$$\Psi_\alpha(p_1, p_2) = \langle K_{q,\alpha}(p_2)K_{\alpha,q}(p_1)|\tilde{\pi}(P)\rangle. \tag{85}$$

This relation defines $\Psi_\alpha(p_1, p_2)$ in the domain $\infty > p_1 > p_2 > -\infty$. Continuation into the whole plane $-\infty < p_1, p_2 < \infty$ provided by the Faddeev–Zamolodchikov commutation relations (8) and (10) yields

$$\begin{aligned}\Psi_\alpha(p_2, p_1) &= \sum_{\beta\neq q} S_{qq}^{\beta\alpha}(\theta_1 - \theta_2)\Psi_\beta(p_1, p_2) \\ &= S_3(\theta_1 - \theta_2)\,\Psi_\alpha(p_1, p_2) + S_2(\theta_1 - \theta_2) \sum_{\beta\neq\alpha,q} \Psi_\beta(p_1, p_2).\end{aligned} \tag{86}$$

Then, equation (84) takes the form

$$\begin{aligned}&[\omega(p_1) + \omega(p_2) - \tilde{E}(P)]\Psi_\alpha(p_1, p_2) = \tilde{E}_{\text{vac}}\Psi_\alpha(p_1, p_2) + \frac{h}{2}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}p_1'\,\mathrm{d}p_2'}{(2\pi)^2} \\ &\cdot \exp[\mathrm{i}x(p_1' + p_2' - p_1 - p_2)] \sum_{\beta=1}^{q-1} \langle K_{q,\alpha}(p_2)K_{\alpha,q}(p_1)|\sigma_q(0)|K_{q,\beta}(p_1')K_{\beta,q}(p_2')\rangle \\ &\times\Psi_\beta(p_1', p_2').\end{aligned} \tag{87}$$

The matrix element in the integral kernel of this equation is the generalized formfactor of the magnetization operator in the momentum representation. It is well known that such formfactors have the so-called *kinematic singularities* at coinciding in- and out-momenta of particles. Let us extract a part of these singularities which are contained in the disconnected 'direct propagation' terms[3]:

$$\begin{aligned}&\langle K_{q,\alpha}(p_2)K_{\alpha,q}(p_1)|\sigma_q(0)|K_{q,\beta}(p_1')K_{\beta,q}(p_2')\rangle = \mathcal{G}_{\alpha\beta}(p_2, p_1|p_1', p_2') \\ &+4\pi^2\langle\sigma_q\rangle_q \left[\delta_{\alpha_\beta}\delta(p_1 - p_1')\delta(p_2 - p_2') + S_{qq}^{\beta\alpha}(\theta_1' - \theta_2')\delta(p_1 - p_2')\delta(p_2 - p_1')\right].\end{aligned} \tag{88}$$

Substitution of the second line of the above formula into the right-hand side of (87) yields

$$\begin{aligned}&\tilde{E}_{\text{vac}}\Psi_\alpha(p_1, p_2) + 4\pi^2 h\langle\sigma_q\rangle_q \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}p_1'\mathrm{d}p_2'}{(2\pi)^2}\delta(p_1 - p_1')\delta(p_2 - p_2')\Psi_\alpha(p_1', p_2') \\ &\cdot \exp[\mathrm{i}x(p_1 + p_2 - p_1' - p_2')] = \left(\tilde{E}_{\text{vac}} + h\langle\sigma_q\rangle_q \int_{-\infty}^{\infty} \mathrm{d}x\right)\Psi_\alpha(p_1, p_2).\end{aligned} \tag{89}$$

While deriving the left-hand side of (89), we have taken into account symmetry (86). It is natural to expect that the two divergent terms in the brackets in the right-hand side of (89) cancel each other in the thermodynamic limit. In fact, in a finite system of length L, the shift of the vacuum energy in the presence of a positive magnetic field h is (in the two-quark approximation) $\tilde{E}_{\text{vac}} = -hL\langle\sigma_q\rangle_q$, and the integral $\int \mathrm{d}x$ should produce the length L of the system.

After cancellation of the infinite terms described above, the right-hand side of (87) becomes well defined in the thermodynamic limit, and we can safely perform in it integration

[3] A detailed analysis of disconnected terms in the integrals containing formfactors in integrable quantum field theories in a finite volume has been done by Pozsgay and Takács [19].

in x. Then equation (87) takes the final form in the variables $p = (p_1 - p_2)/2$, $p' = (p_1' - p_2')/2$:

$$[\omega(P/2+p) + \omega(P/2-p) - \tilde{E}(P)]\Phi_\alpha(p;P) = \frac{h}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}p'}{2\pi}\sum_{\beta=1}^{q-1} G_{\alpha\beta}(p|p';P)\,\Phi_\beta(p';P). \quad (90)$$

Here the meson wavefunction $\Phi_\alpha(p;P)$ and the kernel $G_{\alpha\beta}(p|p';P)$ are defined as

$$\Psi_\alpha(p_1,p_2) = 2\pi\delta(p_1+p_2-P)\Phi_\alpha(p_1-P/2;P), \quad (91)$$

$$G_{\alpha\beta}(p|p';P) = \mathcal{G}_{\alpha\beta}(P/2-p,\, P/2+p|P/2+p',\, P/2-p'). \quad (92)$$

The transformation law for the function $\Phi_\alpha(p;P)$ under the reflection $p \to -p$ can be read from (86) and (91).

Equation (90) gives generalization of the IFT Bethe–Salpeter equation to the q-state PFT with $2 < q \leqslant 4$. Though equation (90) and its derivation look very similar to those in the IFT (see section 3 and equation (3.11) in [2]), an important difference should be pointed out. Derivation of the Bethe–Salpeter equation in the IFT is based on the free-fermionic basis, since the IFT describes noninteracting particles at $h = 0$. In contrast, the quarks in the PFT at $q > 2$ and $h = 0$ strongly interact at small distances $\sim m^{-1}$. The basis states (82), which diagonalize the PFT zero-field Hamiltonian $\hat{\mathcal{H}}_0$, can be treated as its n-quark 'stationary scattering states'. Corresponding wavefunctions in the coordinate representation look like a superposition of plane waves only if the distances between quarks are large compared with the interaction radius. In particular, the wavefunction for the basis state $|K_{q,\alpha}(p_1)K_{\alpha,q}(p_2)\rangle$ reads

$$\psi_\beta(x_1,x_2) = \delta_{\alpha\beta}\exp[\mathrm{i}(p_1x_1+p_2x_2)] + S_{qq}^{\alpha\beta}(\theta_1-\theta_2)\exp[\mathrm{i}(p_2x_1+p_1x_2)], \quad (93)$$

if $x_2 - x_1 \gg m^{-1}$. Here $p_1 > p_2$ is implied for the basis state, and x_1 and x_2 denote the spacial coordinates of the left and right quarks, respectively, $-\infty < x_1 < x_2 < \infty$. However, in the interaction region $0 < x_2 - x_1 \lesssim m^{-1}$, the motion corresponding to the state $|K_{q,\alpha}(p_1)K_{\alpha,q}(p_2)\rangle$ is much more complicated, and it 'cannot be treated in terms of the wavefunction of a finite number of variables (because the virtual pair creation is possible)' [20]. Therefore, one should not understand the term 'the two-quark approximation' in a literal sense in the q-state PFT with $q \neq 2$.

The kernel $G_{\alpha\beta}(p|p';P)$ of the Bethe–Salpeter equation (90) is simply related to the matrix element of the magnetization operator between the two-kink states in the rapidity basis:

$$\langle K_{q,\alpha}(\theta_2)K_{\alpha,q}(\theta_1)|\sigma_q(0)|K_{q,\beta}(\theta_1')K_{\beta,q}(\theta_2')\rangle. = [\omega(p_1)\omega(p_2)\omega(p_1')\omega(p_2')]^{1/2}\langle K_{q,\alpha}(p_2)K_{\alpha,q}(p_1)|\sigma_q(0)|K_{q,\beta}(p_1')K_{\beta,q}(p_2')\rangle. \quad (94)$$

The matrix element can be expressed with the help of the crossing relations in terms of the four-kink elementary formfactor of the magnetization operator:

$$F^{\sigma_q}_{\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4}(\theta_1,\theta_2,\theta_3,\theta_4) = \langle 0_{\alpha_0}|\sigma_q(0)|K_{\alpha_0,\alpha_1}(\theta_1)K_{\alpha_1,\alpha_2}(\theta_2)K_{\alpha_2,\alpha_3}(\theta_3)K_{\alpha_3,\alpha_4}(\theta_4)\rangle, \quad (95)$$

with $\alpha_4 = \alpha_0$. Formfactors are the central objects in the formfactor bootstrap approach in the two-dimensional integrable quantum field theories [17, 21, 22]. The n-particle formfactors are subject to a set of equations (axioms), which often allow one to calculate them exactly (for a review see [22]). Unfortunately, the explicit expressions for all n-kink formfactors in the q-state PFT are known only in the Ising case $q = 2$. Delfino and Cardy [23] obtained the two-kink formfactors $\langle 0_\alpha|\sigma_\gamma(0)|K_{\alpha,\beta}(\theta_1)K_{\beta,\alpha}(\theta_2)\rangle$ of the magnetization operators for $q = 3, 4$.

Calculation of the four-kink formfactors (95) would be crucial for determining the kernel $G_{\alpha\beta}(p|p'; P)$ of equation (90). This kernel should be singular at $p = \pm p'$ due to the kinematic singularities of the function $\mathcal{G}_{\alpha\beta}(p_2, p_1|p_1', p_2')$. We expect that the leading singularity of $G_{\alpha\beta}(p|p'; P)$ at $p \to p'$ has the form

$$\delta_{\alpha\beta}(\langle\sigma_q\rangle_q - \langle\sigma_q\rangle_\alpha)\left[\frac{1}{(p-p'+\mathrm{i}0)^2} + \frac{1}{(p-p'-\mathrm{i}0)^2}\right]. \tag{96}$$

This term should produce after the Fourier transform the long-range attractive potential $\Delta\mathcal{E}|x|$ between the quarks. On the other hand, the regular part of $G_{\alpha\beta}(p|p'; P)$ at $p = \pm p'$ should describe the change in their short-range interaction induced by the magnetic field.

As we know from the perturbative solution of the Bethe–Salpeter equation in the IFT [2, 11], the regular (short-range) part of the integral kernel contributes to the meson masses $M_n(h)$ only in the *second* order in h. Physically, the additional factor h reflects that the two quarks bound in the meson spend at $h \to 0$ almost all the time at large distances. They only rarely appear in the scattering region $x_2 - x_1 \sim m^{-1}$, where the h-order correction to the short-range interaction due to the regular part of $G_{\alpha\beta}(p|p'; P)$ should be taken into account.

6. Conclusion

We extended the heuristic perturbative approach, which was originally developed [2, 9, 10] for calculation of the meson masses in the weak confinement regime in the IFT, to the q-state PFT with $2 < q \leqslant 4$ in the presence of a weak magnetic field h. Though the latter model is integrable at $h = 0$, the kinks ('quarks') remain to be interacting particles at $h = 0$. We have calculated the masses $M_n(h)$ of the mesons in the PFT at $T < T_c$ in the leading order in the weak magnetic field $|h| \to 0$ both in the low energy and semiclassical cases. The mesons with nonzero topological charge were predicted for the 3-state PFT in the ordered phase at $h < 0$. The Bethe–Salpeter equation is derived for the q-state PFT with $2 < q \leqslant 4$, which generalizes the analogous equation known in the IFT. This equation could provide a more firm basis for the theory, if the explicit expressions for the four-kink formfactors of the magnetization operator would be found.

After the first version of this paper had appeared as a preprint, Lepori, Tóth and Delfino [24] presented the results of their numerical investigations of the particle spectra in the 3-state PFT in a wide range of magnetic fields h and temperatures τ by means of the truncated conformal space approach (TCSA) [25]. They confirmed the qualitative picture of confinement developed in [5], and were able to partly confirm our analytical predictions (36) and (64) for the low-energy part of the meson spectra at $h \to +0$. Reported in [24], the magnetic field dependence of the masses of five lightest even ($i = 0$) and odd ($i = 1$) mesons at moderately large magnetic fields was described by the formula

$$M_n^{(i)} = 2m + c_n^{(i)} h^\alpha, \tag{97}$$

with

$$\alpha \approx 0.7, \qquad \frac{c_1^{(1)}}{c_1^{(0)}} \approx 2. \tag{98}$$

These values are in reasonable agreement with the numbers

$$\alpha = \frac{2}{3}, \qquad \frac{c_1^{(1)}}{c_1^{(0)}} = \frac{z_1}{z_1'} \approx 2.3, \tag{99}$$

following from (36) and (64). However, for a complete numerical verification one needs to increase the accuracy of the TCSA calculations at small $|h|$.

The obtained results could be developed in several directions. First, it is straightforward to extend them to the wide class of models exhibiting confinement, which are integrable but not free at zero 'magnetic field', see [8]. Regarding the PFT, one can try to find from the Bethe–Salpeter equation (90) corrections to the meson masses at small $|h|$, and to study the decay mechanisms for unstable mesons in the higher orders in $|h|$.

It is remarkable [2] that in the IFT, the Bethe–Salpeter equation reproduces with reasonable accuracy the mesons masses not only in the limit $h \to 0$, but also at finite, and even at large values of the magnetic field h. If this situations also holds for the PFT, equation (90) could be useful for nonperturbative calculations of the meson spectra in the PFT at finite magnetic fields. To achieve progress in all these directions, it is essential to find explicit expressions for the n-kink formfactors of the magnetization operator in the PFT.

One more interesting open problem is to determine in the PFT the masses of 'baryons' consisting of three quarks [5].

We close with the following remark. For the sake of simplicity, we calculated in sections 3 and 4 the energy of a meson which has zero momentum, i.e. analyzed the problem in the meson rest frame. It is straightforward to modify calculations to the case of a generic frame, and to check (in the leading order in h) that the meson dispersion law $E(P)$ really has the relativistic form (22), as one should expect.

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